JACQUES TITS' MOTIVIC MEASURE

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Abstract. Making use of the recent theory of noncommutative motives, we construct a new motivic measure, which we call the *Tits' motivic measure*. As a first application, we prove that two Severi-Brauer varieties (or more generally twisted Grassmannian varieties), associated to central simple algebras of period 2, have the same Grothendieck class if and only if they are isomorphic. As a second application, we show that if two Severi-Brauer varieties, associated to central simple algebras of period $2, 3, 4, 5$ or 6, have the same Grothendieck class, then they are necessarily birational. As a third application, we prove that two quadric hypersurfaces (or more generally involution varieties), associated to quadratic forms of degree 6, have the same Grothendieck class if and only if they are isomorphic. This latter result also holds for products of such quadrics. Finally, as a fourth application, we show in certain cases that two products of conics have the same Grothendieck class if and only if they are isomorphic; this refines a result of Kollár.

1. Introduction

Motivic measures. Let k be a base field and $\mathrm{Var}(k)$ the category of *varieties*, *i.e.* reduced separated k-schemes of finite type. The *Grothendieck ring of varieties* $K_0\mathrm{Var}(k)$, introduced in the sixties in a letter from Grothendieck to Serre, is defined as the quotient of the free abelian group on the set of isomorphism classes of varieties $[X]$ by the "cut-and-paste" relations $[X] = [Y] + [X\backslash Y]$, where Y is a closed subvariety of X. The multiplication law is induced by the product of varieties. Although very important, the structure of this ring is still nowadays poorly understood. For example, $K_0\mathrm{Var}(k)$ is not a domain (see Kollár [13], Naumann [24] and Poonen [27]) and the Grothendieck class $[\mathbb{A}^1]$ of the affine line $\mathbb{A}^1$ is a zero divisor (see Borisov [7]). In order to capture some of the flavor of the Grothendieck ring of varieties, several *motivic measures*, *i.e.* ring homomorphisms $\mu\colon K_0\mathrm{Var}(k) \to R$, have been built. Here are some classical examples:

(i) when k is finite, the assignment $X \mapsto \#X(k)$ gives rise to the counting motivic measure $\mu_\#\colon K_0\mathrm{Var}(k) \to \mathbb{Z}$;
(ii) when $k = \mathbb{C}$, the assignment $X \mapsto \chi_c(X) := \sum_n (-1)^n \dim H^n_c(X^{\mathrm{an}};\mathbb{Q})$ gives rise to the Euler characteristic motivic measure $\chi_c\colon K_0\mathrm{Var}(k) \to \mathbb{Z}$;
(iii) when $\mathrm{char}(k) = 0$, the assignment $X \mapsto \mathrm{P}_X(u) := \sum_n \dim H^n_{dR}(X)u^n$ gives rise to the Poincaré characteristic motivic measure $\mu_{\mathrm{P}}\colon K_0\mathrm{Var}(k) \to \mathbb{Z}[u]$;
(iv) when $\mathrm{char}(k) = 0$, the assignment $X \mapsto \mathrm{H}_X(u,v) := \sum_{p,q} h^{p,q}(X)u^pv^q$ gives rise to the Hodge characteristic motivic measure $\mu_{\mathrm{H}}\colon K_0\mathrm{Var}(k) \to \mathbb{Z}[u,v]$.

In this article we introduce a new motivic measure μ_{T}, which we name the *Tits' motivic measure*. Making use of it, we establish several new structural properties of the Grothendieck ring of varieties; consult §2 for details.

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Twisted projective homogeneous varieties. Let k be a base field and $\Gamma := \mathrm{Gal}(k_{\mathrm{sep}}/k)$ its absolute Galois group. Given a split semi-simple algebraic group G over k, a parabolic subgroup $P \subset G$, and a 1-cocycle $\gamma\colon \Gamma \to G(k_{\mathrm{sep}})$, we can construct the projective homogeneous variety $\mathcal{F} := G/P$ as well as its twisted form ${}_\gamma\mathcal{F}$. Let $\widetilde{G}$ and $\widetilde{P}$ be the universal covers of G and P, $R(\widetilde{G})$ and $R(\widetilde{P})$ the associated representation rings, $n(\mathcal{F})$ the index $[W(\widetilde{G}) : W(\widetilde{P})]$ of the Weyl groups, $\widetilde{Z}$ the center of $\widetilde{G}$, and finally Ch the character group $\mathrm{Hom}(\widetilde{Z}, \mathbb{G}_m)$. As proved by Steinberg in [29] (see also Panin [26, §12.5-12.8]), the ring $R(\widetilde{P})$ admits a canonical Ch-homogeneous basis $\rho_i, 1 \leq i \leq n(\mathcal{F})$, over $R(\widetilde{G})$. Let us denote by A_i the Tits' central simple k-algebra associated to ρ_i; consult [18, §27][36] for details.

Example 1.1 (Severi-Brauer varieties). Let G be the projective general linear group PGL_n. In this case, we have $\widetilde{G} = SL_n$. Consider the following parabolic subgroup

$$\widetilde{P} := \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a \cdot \det(c) = 1 \} \subset SL_n \quad a \in k^\times \quad c \in GL_{n-1}\,.$$

The projective homogeneous variety $\mathcal{F} := G/P \simeq \widetilde{G}/\widetilde{P}$ is the projective space $\mathbb{P}^{n-1}$ and $n(\mathcal{F}) = n$. Given a 1-cocycle $\gamma\colon \Gamma \to PGL_n(k_{\mathrm{sep}})$, let A be the corresponding central simple k-algebra of degree n. Under these notations, the twisted form ${}_\gamma\mathbb{P}^{n-1}$ is the Severi-Brauer variety $\mathrm{SB}(A)$ associated to A and the Tits' algebras are $k, A, A^{\otimes 2}, \ldots, A^{\otimes (n-1)}$.

Example 1.2 (Twisted Grassmannian varieties). Let $G = PGL_n$. Choose an integer $1 \leq d < n$ and consider the following parabolic subgroup

$$\widetilde{P} := \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid \det(a) \cdot \det(c) = 1 \} \subset SL_n \quad a \in GL_d \quad c \in GL_{n-d}\,.$$

The projective homogeneous variety $\mathcal{F} := G/P \simeq \widetilde{G}/\widetilde{P}$ is the Grassmannian variety $\mathrm{Gr}(d)$ and $n(\mathcal{F}) = \binom{n}{d}$. Given a 1-cocycle $\gamma\colon \Gamma \to PGL_n(k_{\mathrm{sep}})$, let A be the corresponding central simple k-algebra. Under these notations, the twisted form ${}_\gamma\mathrm{Gr}(d)$ is the twisted Grassmannian variety $\mathrm{Gr}(d; A)$ associated to A and the Tits' algebras (without repetitions) are $k, A, A^{\otimes 2}, \ldots, A^{\otimes (n-d)d}$.

Remark 1.3. When $d = 1$, Example 1.2 reduces to Example 1.1.

Example 1.4 (Quadric hypersurfaces). Let G be the special orthogonal group SO_n with $n \geq 3$. In this case, we have $\widetilde{G} = \mathrm{Spin}_n$. Consider the action of G on $\mathbb{P}^{n-1}$ given by projective linear transformations. We write $P \subset G$ for the stabilizer of the point $[\mathbf{1} : 0 : \cdots : 0]$ and $\widetilde{P}$ for the pre-image of P in $\widetilde{G}$. The projective homogeneous variety $\mathcal{F} := G/P \simeq \widetilde{G}/\widetilde{P}$ is the following smooth quadric hypersurface

$$Q := (x_1 y_1 + \cdots + x_{[\frac{n}{2}]} y_{[\frac{n}{2}]} + z^2 = 0) \subset \mathbb{P}^{n-1} \qquad n \text{ odd}$$
$$Q := (x_1 y_1 + \cdots + x_{\frac{n}{2}} y_{\frac{n}{2}} = 0) \subset \mathbb{P}^{n-1} \qquad n \text{ even}$$

and the index $n(\mathcal{F})$ is equal to $n-1$, resp. n, when n is odd, resp. even. Given a 1-cocycle $\gamma\colon \Gamma \to SO_n(k_{\mathrm{sep}})$, let q be the corresponding non-degenerate quadratic form of dimension n. We write $C_0(q)$ for the associated even Clifford algebra. When n is odd, $C_0(q)$ is a central simple k-algebra. When n is even we assume that the discriminant $\delta(q) \in k^\times$ is *trivial*, i.e. $\delta(q) \in (k^\times)^2$. In this latter case, $C_0(q) \simeq C_0^+(q) \times C_0^-(q)$ decomposes into two isomorphic central simple k-algebras. Under the above assumptions, the twisted form ${}_\gamma Q$ is the smooth quadric hypersurface

$Q_q \subset \mathbb{P}^{n-1}$ associated to q and the Tits's algebras (without repetitions) are k and $C_0(q)$ when n is odd and k, $C_0^+(q)$, and $C_0^-(q)$, when n is even.

Example 1.5 (Twisted quaternion projective spaces). Let G be the projective symplectic group PSp_n with n even. In this case, we have $\widetilde{G} = Sp_n$. Consider the parabolic subgroup $\widetilde{P} := Sp_2 \times Sp_{n-2}$ of Sp_n. The projective homogeneous variety $\mathcal{F} := G/P \simeq \widetilde{G}/\widetilde{P}$ is the quaternion projective space $\mathrm{HP}^{\frac{n}{2}-1}$ and $n(\mathcal{F}) = \frac{n}{2}$. Given a 1-cocycle $\gamma\colon \Gamma \to PSp_n(k_{\mathrm{sep}})$, let $(A, *)$ be the corresponding central simple k-algebra of degree n with involution of symplectic type. Under these notations, the twisted form ${}_\gamma\mathrm{HP}^{\frac{n}{2}-1}$ is the twisted quaternion projective space $\mathrm{HP}(A, *)$ associated to $(A, *)$ and the Tits' algebras (without repetitions) are k and A.

Example 1.6 (Involution varieties). Let G be the projective special orthogonal group PSO_n with n even. In this case, we have $\widetilde{G} = \mathrm{Spin}_n$. Similarly to Example 1.4, consider the smooth quadric hypersurface $Q := (x_1y_1 + \cdots + x_{\frac{n}{2}}y_{\frac{n}{2}} = 0) \subset \mathbb{P}^{n-1}$. Given a 1-cocycle $\gamma\colon \Gamma \to PSO_n(k_{\mathrm{sep}})$, let $(A, *)$ be the corresponding central simple k-algebra of degree n with involution of orthogonal type. We write $C_0(A, *)$ for the associated even Clifford algebra and assume that the discriminant $\delta(A, *)$ is trivial. In this latter case, $C_0(A, *) \simeq C_0^+(A, *) \times C_0^-(A, *)$ decomposes into two central simple k-algebras. Under the above assumptions, the twisted form ${}_\gamma Q$ is the involution variety $\mathrm{Iv}(A, *)$ associated to $(A, *)$ and the Tits' algebras (without repetitions) are k, A, $C_0^+(A, *)$, and $C_0^-(A, *)$.

Remark 1.7. When $(A, *)$ is *split*, *i.e.* isomorphic to $(M_n(k), *_q)$ with $*_q$ the adjoint involution associated to a quadratic form q, Example 1.6 reduces to Example 1.4.

Statement of results. Let k be a base field of characteristic zero. In what follows, we denote by $K_0\mathrm{Var}(k)^{\mathrm{tw}}$ the smallest subring of $K_0\mathrm{Var}(k)$ containing the Grothendieck classes $[{}_\gamma\mathcal{F}]$ of all twisted projective homogeneous varieties (for all possible choices of G, P and γ). Consider the Brauer group $\mathrm{Br}(k)$ of k, the group ring $\mathbb{Z}[\mathrm{Br}(k)]$ of $\mathrm{Br}(k)$, and the following quotient ring

$$R_{\mathrm{T}}(k) := \mathbb{Z}[\mathrm{Br}(k)]/\langle [k] + [B \otimes C] - [B] - [C] \,|\, (\mathrm{ind}(B), \mathrm{ind}(C)) = 1 \rangle \,,$$

where B and C are arbitrary central simple k-algebras with coprime indexes. Recall from [8, Prop. 4.5.16] the p-primary decomposition $\mathrm{Br}(k) = \oplus_p \mathrm{Br}(k)\{p\}$ of the Brauer group. Note that in the particular case where every element of $\mathrm{Br}(k)$ is of p-primary torsion, the quotient ring $R_{\mathrm{T}}(k)$ reduces to the group ring of $\mathrm{Br}(k)\{p\}$. Under the above notations, our main result is the following:

Theorem 1.8. *The assignment ${}_\gamma\mathcal{F} \mapsto \sum_{i=1}^{n(\mathcal{F})}[A_i]$ gives rise to a motivic measure $\mu_{\mathrm{T}}\colon K_0\mathrm{Var}(k)^{\mathrm{tw}} \to R_{\mathrm{T}}(k)$, which we name the* Tits' motivic measure*.*

Intuitively speaking, Theorem 1.8 shows that the (noncommutative) Tits' algebras associated to a twisted projective homogeneous variety are preserved by the (geometric) "cut-and-paste" relations. The proof of this result makes essential use of the recent theory of noncommutative motives (see §3). Indeed, by construction, μ_{T} is the restriction of a certain motivic measure defined on the whole Grothendieck ring of varieties and with values in the Grothendieck ring of the additive symmetric monoidal category of noncommutative Chow motives; consult §4 for details.

Note that $R_{\mathrm{T}}(k)$ comes equipped with the augmentation $\sum_{i=1}^m b_i[B_i] \mapsto \sum_{i=1}^m b_i$. By pre-composing it with μ_{T}, we obtain the following motivic measure:

$$K_0\mathrm{Var}(k)^{\mathrm{tw}} \longrightarrow \mathbb{Z} \qquad [{}_\gamma\mathcal{F}] \mapsto n(\mathcal{F})\,. \tag{1.9}$$

Corollary 1.10. *Let ${}_\gamma\mathcal{F}$ and ${}_{\gamma'}\mathcal{F}'$ be two twisted projective homogeneous varieties. If $[{}_\gamma\mathcal{F}] = [{}_{\gamma'}\mathcal{F}']$ in $K_0\mathrm{Var}(k)$, then $n(\mathcal{F}) = n(\mathcal{F}')$.*

Similarly to $R_{\mathrm{T}}(k)$, consider the following quotient semi-ring:

$$R^+_{\mathrm{T}}(k) := \mathbb{N}[\mathrm{Br}(k)]/\{[k] + [B \otimes C] = [B] + [C] \,|\, (\mathrm{ind}(B), \mathrm{ind}(C)) = 1\}\,.$$

Proposition 1.11. (i) *The homomorphism $R^+_{\mathrm{T}}(k) \to R_{\mathrm{T}}(k)$ is injective;*
(ii) *The assignment $\sum_{i=1}^m b_i[B_i] \mapsto \langle [B_1], \ldots, [B_m]\rangle$, where $\langle [B_1], \ldots, [B_m]\rangle$ stands for the subgroup of $\mathrm{Br}(k)$ generated by the Brauer classes $[B_1], \ldots, [B_m]$, gives rise to a well-defined surjective map $R^+_{\mathrm{T}}(k) \twoheadrightarrow \{\text{subgroups of } \mathrm{Br}(k)\}$.*

Roughly speaking, Proposition 1.11 shows that $R^+_{\mathrm{T}}(k)$ is the "positive cone" of $R_{\mathrm{T}}(k)$ and that this semi-ring encodes all the information concerning the subgroups of $\mathrm{Br}(k)$. By combining Proposition 1.11 with Theorem 1.8, we obtain the result:

Corollary 1.12. *Let ${}_\gamma\mathcal{F}$ and ${}_{\gamma'}\mathcal{F}'$ be two twisted projective homogeneous varieties with Tits' central simple k-algebras $A_1, \ldots, A_{n(\mathcal{F})}$ and $A'_1, \ldots, A'_{n(\mathcal{F}')}$, respectively. If $[{}_\gamma\mathcal{F}] = [{}_{\gamma'}\mathcal{F}']$ in $K_0\mathrm{Var}(k)$, then $\langle [A_1], \ldots, [A_{n(\mathcal{F})}]\rangle = \langle [A'_1], \ldots, [A'_{n(\mathcal{F}')}]\rangle$.*

Remark 1.13. An alternative proof of Corollary 1.12 can be obtained in two steps:
(i) Firstly, as proved by Larsen-Lunts in [20, Thm. 2.3], if two varieties have the same Grothendieck class, then they are stably birational. Consequently, if $[{}_\gamma\mathcal{F}] = [{}_{\gamma'}\mathcal{F}']$ in $K_0\mathrm{Var}(k)$, we then conclude that ${}_\gamma\mathcal{F}$ and ${}_{\gamma'}\mathcal{F}'$ are stably birational. Note that these two twisted forms have the same dimension.
(ii) Secondly, as proved by Merkurjev-Tignol[1] in [23, Thm. B], the kernel of the base-change homomorphism $\mathrm{Br}(k) \to \mathrm{Br}(k({}_\gamma\mathcal{F}))$, where $k({}_\gamma\mathcal{F})$ stands for the function field of ${}_\gamma\mathcal{F}$, is given by the subgroup $\langle [A_1], \ldots, [A_{n(\mathcal{F})}]\rangle$. Therefore, if ${}_\gamma\mathcal{F}$ and ${}_{\gamma'}\mathcal{F}'$ are stably birational and have the same dimension, it can be shown that $\langle [A_1], \ldots, [A_{n(\mathcal{F})}]\rangle = \langle [A'_1], \ldots, [A'_{n(\mathcal{F}')}]\rangle$.

As mentioned above, our proof of Corollary 1.12 is intrinsically different. In the spirit of Bondal-Orlov [5, 6], we study the twisted projective homogeneous varieties via their derived categories and the associated noncommutative (Chow) motives. This "noncommutative viewpoint" enables the construction of the new motivic measure μ_{T}, from which Corollary 1.12 stems out as a simple byproduct.

2. Applications

Severi-Brauer varieties. In this subsection, we follow the notations of Example 1.1. Given a central simple k-algebra A, we write $\deg(A)$ for its degree, $\mathrm{ind}(A)$ for its index, and $\mathrm{per}(A)$ for its period.

Theorem 2.1. *Given central simple k-algebras A and A', consider the conditions:*
(i) *we have $[\mathrm{SB}(A)] = [\mathrm{SB}(A')]$ in $K_0\mathrm{Var}(k)$;*
(ii) *we have $\mu_{\mathrm{T}}([\mathrm{SB}(A)]) = \mu_{\mathrm{T}}([\mathrm{SB}(A')])$ in $R_{\mathrm{T}}(k)$;*
(iii) *we have $\deg(A) = \deg(A')$ and $\langle [A]\rangle = \langle [A']\rangle$;*
(iv) *we have an isomorphism $A \simeq A'$;*
(v) *we have $\mathrm{SB}(A) \simeq \mathrm{SB}(A')$ in $\mathrm{Var}(k)$.*

We have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftarrow$ (iv) $\Leftrightarrow$ (v) $\Rightarrow$ (i). *Whenever* $\mathrm{per}(A) = \mathrm{per}(A') = 2$, *we have moreover the implication* (iii) $\Rightarrow$ (iv).

[1] In the case of Severi-Brauer varieties, consult also the pioneering work of Amitsur [2].

Note that $\langle [A] \rangle = \langle [A'] \rangle$ if and only if $[A'] = m[A]$ in $\mathrm{Br}(k)$ for some integer m coprime to $\mathrm{per}(A)$. Note also that in the case where A and A' have period 2, all the conditions (i)-(v) of Theorem 2.1 are equivalent. Therefore, two Severi-Brauer varieties, associated to central simple k-algebras of period 2, have the same Grothendieck class in $K_0\mathrm{Var}(k)$ if and only if they are isomorphic! To the best of the author's knowledge, this result is new in the literature.

Example 2.2 (Real Severi-Brauer varieties). Recall that the Brauer group $\mathrm{Br}(\mathbb{R})$ of $\mathbb{R}$ is the cyclic group of order 2; the quaternions $\mathbb{H}$ are the only non-trivial central division $\mathbb{R}$-algebra. Making use of Theorem 2.1, we hence conclude that non-isomorphic real Severi-Brauer varieties have distinct Grothendieck classes in $K_0\mathrm{Var}(k)$. Furthermore, the Tits' motivic measure μ_{T} becomes a complete invariant when restricted to real Severi-Brauer varieties. Recall from §1 that $R_{\mathrm{T}}(\mathbb{R})$ reduces to the group ring of the cyclic group of order 2.

Remark 2.3. Example 2.2 holds more generally whenever $\mathrm{Br}(k) = {}_2\mathrm{Br}(k)$.

Example 2.4 (Conics). Let A be a central simple k-algebra of degree 2 (and hence of period 2). In this particular case, A is isomorphic to a quaternion algebra (a, b), with $a, b \in k^\times$, and the associated Severi-Brauer variety $\mathrm{SB}(A)$ is given by the (smooth) conic $C(a, b) := (ax^2 + by^2 - z^2 = 0) \subset \mathbb{P}^2$; see [8, §1]. Making use of Theorem 2.1, we hence conclude that non-isomorphic conics $C(a, b)$ have distinct Grothendieck classes in $K_0\mathrm{Var}(k)$. Furthermore, the Tits' motivic measure μ_{T} becomes a complete invariant when restricted to smooth conics.

Example 2.5 (Rational coefficients). When $k = \mathbb{Q}$, the group ${}_2\mathrm{Br}(\mathbb{Q})$ is infinite. Following Example 2.4, we have the infinite family of distinct Grothendieck classes

$$[C(-1, p)] := [(-x^2 + py^2 - z^2 = 0)] \qquad p \equiv 3 \pmod 4$$

in $K_0\mathrm{Var}(\mathbb{Q})$. Note that the conics $C(-1, p)$ are not birational to $\mathbb{P}^1$.

Remark 2.6 (Products of conics). Let k be a number field or the function field of an algebraic surface over $\mathbb{C}$. Given elements $a_1, b_1, a_2, b_2, a_1', b_1', a_2', b_2' \in k$, Kollár proved in [13, Thm. 2] that the following three conditions are equivalent:[2]

(i) we have $[C(a_1, b_1) \times C(a_2, b_2)] = [C(a_1', b_1') \times C(a_2', b_2')]$ in $K_0\mathrm{Var}(k)$;
(ii) $C(a_1, b_1) \times C(a_2, b_2)$ is birational to $C(a_1', b_1') \times C(a_2', b_2')$;
(iii) we have $\langle [(a_1, b_1)], [(a_2, b_2)] \rangle = \langle [(a_1', b_1')], [(a_2', b_2')] \rangle$.

Note that the equivalence (i) $\Leftrightarrow$ (iii) implies that the Tits' motivic measure μ_{T} becomes a complete invariant when restricted to products of conics.

Corollary 2.25 below shows, in certain cases, that the word "birational" of item (ii) of Remark 2.6 can be replaced by the word "isomorphic". This refines Kollár's result [13, Thm. 2]. In what concerns products of two Severi-Brauer varieties (of arbitrary dimension), we have the following result:

Proposition 2.7. *Let A, A', A'', A''' be central simple k-algebras of period 2, with $\deg(A) = \deg(A')$ and $\deg(A'') = \deg(A''')$. If $[\mathrm{SB}(A) \times \mathrm{SB}(A')] = [\mathrm{SB}(A'') \times \mathrm{SB}(A''')]$ in $K_0\mathrm{Var}(k)$, then $\mathrm{SB}(A)$ (or $\mathrm{SB}(A')$) is isomorphic to $\mathrm{SB}(A'')$ or $\mathrm{SB}(A''')$.*

[2] Using inductive arguments, Kollár considered more generally finite products of conics. Later, Hogadi [9] removed the restrictions on the base field k.

(Stable) birationality. Let A and A' be two central simple k-algebras with the same degree. Recall from [8, Rk. 5.4.3] that if $\langle[A]\rangle = \langle[A']\rangle$, then the Severi-Brauer varieties $\mathrm{SB}(A)$ and $\mathrm{SB}(A')$ are stably birational. Making use of implication (i) $\Rightarrow$ (iii) of Theorem 2.1, we hence obtain automatically the following result:

Corollary 2.8. *If $[\mathrm{SB}(A)] = [\mathrm{SB}(A')]$ in $K_0\mathrm{Var}(k)$, then the Severi-Brauer varieties $\mathrm{SB}(A)$ and $\mathrm{SB}(A')$ are stably birational.*

Recall from Amitsur [2, §9] the following deep conjecture linking (noncommutative) algebra with (birational) geometry:

Amitsur's conjecture: If $\langle[A]\rangle = \langle[A']\rangle$, then $\mathrm{SB}(A)$ is birational to $\mathrm{SB}(A')$.

As proved in *loc. cit.*, this conjecture holds whenever k is a *global field*, *i.e.* a finite field extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$, or a *local field*, *i.e.* a finite field extension of $\mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{F}_p((t))$. Making use of the implication (i) $\Rightarrow$ (iii) of Theorem 2.1, we hence obtain automatically the following conditional result:

Corollary 2.9 (Conditional)**.** *If Amitsur's conjecture holds and $[\mathrm{SB}(A)] = [\mathrm{SB}(A')]$ in $K_0\mathrm{Var}(k)$, then $\mathrm{SB}(A)$ is birational to $\mathrm{SB}(A')$.*

Corollary 2.9 suggests that birationality is preserved by the "cut-and-paste" relations[3]. For certain values of the period, we have the following unconditional result:

Proposition 2.10 (Unconditional)**.** *Assume that A and A' have period $2, 3, 4, 5$ or 6; in the case of period 5, we assume moreover that $\deg(A) = \deg(A')$ is even. If $[\mathrm{SB}(A)] = [\mathrm{SB}(A')]$ in $K_0\mathrm{Var}(k)$, then $\mathrm{SB}(A)$ is birational to $\mathrm{SB}(A')$.*

Remark 2.11 (Severi-Brauer surfaces). Let A and A' be central simple k-algebras of degree 3 (and hence of period 3). In this particular case, Hogadi proved in [9, Thm. 1.2], using different arguments, that if $[\mathrm{SB}(A)] = [\mathrm{SB}(A')]$ in $K_0\mathrm{Var}(k)$, then $\mathrm{SB}(A)$ is birational to $\mathrm{SB}(A')$. Note that in Proposition 2.10 we don't impose any restriction on the degree (except being a multiple of 3).

Twisted Grassmannian varieties. In this subsection, we follow the notations of Example 1.2. The following result generalizes Theorem 2.1.

Theorem 2.12. *Given central simple k-algebras A and A' and integers $1 \leq d < \deg(A)$ and $1 \leq d' < \deg(A')$, consider the following five conditions:*

(i) *we have $[\mathrm{Gr}(d; A)] = [\mathrm{Gr}(d'; A')]$ in $K_0\mathrm{Var}(k)$;*
(ii) *we have $\mu_{\mathrm{T}}([\mathrm{Gr}(d; A)]) = \mu_{\mathrm{T}}([\mathrm{Gr}(d'; A')])$ in $R_{\mathrm{T}}(k)$;*
(iii) *we have $\deg(A) = \deg(A')$, $d' = d$ or $d' = \deg(A) - d$, and $\langle[A]\rangle = \langle[A']\rangle$;*
(iv) *we have $d' = d$ or $d' = \deg(A) - d$ and an isomorphism $A \simeq A'$;*
(v) *we have $\mathrm{Gr}(d; A) \simeq \mathrm{Gr}(d'; A')$ in $\mathrm{Var}(k)$.*

We have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftarrow$ (iv) $\Leftrightarrow$ (v) $\Rightarrow$ (i). Whenever $\mathrm{per}(A) = \mathrm{per}(A') = 2$, we have moreover the implication (iii) $\Rightarrow$ (iv).

Note that in the case where A and A' have period 2, all the conditions (i)-(v) of Theorem 2.12 are equivalent. Therefore, two twisted Grassmannian varieties, associated to central simple k-algebras of period 2, have the same Grothendieck class in $K_0\mathrm{Var}(k)$ if and only if they are isomorphic! To the best of the author's knowledge, this result is new in the literature.

Example 2.2 and Remark 2.3 hold *mutatis mutandis* for twisted Grassmannian varieties. Moreover, Proposition 2.7 admits the following generalization:

[3]The converse does not holds in general; see Example 2.16.

Proposition 2.13. *Let A, A', A'', A''' be central simple k-algebras of period 2, with $\deg(A) = \deg(A')$ and $\deg(A'') = \deg(A''')$, and $1 \leq d < \deg(A)$ and $1 \leq d'' < \deg(A'')$ integers. If $[\mathrm{Gr}(d; A) \times \mathrm{Gr}(d; A')]$ agrees with $[\mathrm{Gr}(d''; A'') \times \mathrm{Gr}(d''; A''')]$ in $K_0\mathrm{Var}(k)$, then $\mathrm{Gr}(d; A)$ (or $\mathrm{Gr}(d; A')$) is isomorphic to $\mathrm{Gr}(d''; A'')$ or $\mathrm{Gr}(d''; A''')$.*

Quadric hypersurfaces. We follow the notations/assumptions of Example 1.4.

Theorem 2.14. *Given quadratic forms q and q', consider the five conditions:*
- (i) *we have $[Q_q] = [Q_{q'}]$ in $K_0\mathrm{Var}(k)$;*
- (ii) *we have $\mu_{\mathrm{T}}([Q_q]) = \mu_{\mathrm{T}}([Q_{q'}])$ in $R_{\mathrm{T}}(k)$;*
- (iii) *we have $\dim(q) = \dim(q')$ and an isomorphism $C_0(q) \simeq C_0(q')$, resp. $C_0^+(q) \simeq C_0^+(q')$, of k-algebras in the odd-dimensional, resp. even-dimensional, case;*
- (iv) *the quadratic forms q and q' are* similar, *i.e. $q \simeq a \cdot q'$ for some $a \in k^\times$;*
- (v) *we have $Q_q \simeq Q_{q'}$ in $\mathrm{Var}(k)$.*

We have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftarrow$ (iv) $\Leftrightarrow$ (v) $\Rightarrow$ (i). Whenever $\dim(q) = \dim(q')$ is equal to 3 or 6, we have moreover the implication (iii) $\Rightarrow$ (iv).

Note that in the case where q and q' have dimension 3, all the conditions (i)-(v) of Theorem 2.14 are equivalent. The quadric hypersurface Q_q of a quadratic form of dimension 3 with trivial discriminant is given by the conic associated to the quaternion algebra $C_0(q)$. Therefore, in these cases, the equivalence (i) $\Leftrightarrow$ (v) of Theorem 2.14 reduces to Remark 2.6 (of Theorem 2.1).

Note also that when q and q' have dimension 6, all the conditions (i)-(v) of Theorem 2.14 are equivalent. Recall that, up to similarly, a quadratic form q of dimension 6 with trivial discriminant is given by $\langle a_1, b_1, -a_1b_1, -a_2, -b_2, a_2b_2 \rangle$ with $a_1, b_1, a_2, b_2 \in k^\times$; see [18, §16.4]. These are called *Albert forms*. Thanks to Theorem 2.14, we hence conclude that two of the following smooth quadric hypersurfaces

$$Q_q := (a_1u^2 + b_1v^2 - a_1b_1w^2 - a_2x^2 - b_2y^2 + a_2b_2z^2 = 0) \subset \mathbb{P}^5 \tag{2.15}$$

have the same Grothendieck class in $K_0\mathrm{Var}(k)$ if and only if they are isomorphic! To the best of the author's knowledge, this result is new in the literature.

Example 2.16 (Rational coefficients). When $k = \mathbb{Q}$, we have the following infinite family of distinct Grothendieck classes

$$[Q_q] := [(u^2 + v^2 - w^2 + x^2 - py^2 - pz^2 = 0)] \qquad p \equiv 3 \pmod 4 \tag{2.17}$$

in $K_0\mathrm{Var}(\mathbb{Q})$. In contrast with Example 2.5, note that since all the quadrics Q_q in (2.17) have a rational point, they are birational to $\mathbb{P}^4$; see [14, Thm. 1.11]. Roughly speaking, this shows that in the case of quadric hypersurfaces the Grothendieck class contains much more information than the birational equivalence class.

Surprisingly, the above "rigidity" phenomenon occurs also in the case of products of two smooth quadric hypersurfaces:

Proposition 2.18. *Given quadratic forms q, q', q'', q''' of dimension 6 with trivial discriminant, the following two conditions are equivalent:*
- (i) *we have $[Q_q \times Q_{q'}] = [Q_{q''} \times Q_{q'''}]$ in $K_0\mathrm{Var}(k)$;*
- (ii) *$Q_q \times Q_{q'}$ is isomorphic to $Q_{q''} \times Q_{q'''}$.*

Remark 2.19 (Quadratic forms of dimension 5). As explained in [18, §15.C], the assignment $q \mapsto C_0(q)$ induces a one-to-one correspondence between isometry classes of quadratic forms of dimension 5 with trivial discriminant and isomorphism classes of central simple k-algebras of degree 4 with involution of symplectic type. Recall

that every biquaternion algebra, such as $C_0(q)$, admits an involution of symplectic type. Given biquaternion algebras A and A', there exist then quadratic forms q and q', of dimension 5 with trivial discriminant, such that $C_0(q) \simeq A$ and $C_0(q') \simeq A'$. Whenever $A \not\simeq A'$, we hence conclude from the implication (i) $\Rightarrow$ (iii) of Theorem 2.14 that $[Q_q] \neq [Q_{q'}]$ in $K_0\mathrm{Var}(k)$.

By slightly modifying the arguments of Lewis' work [21], we obtain the following far reaching generalization of Remark 2.19:

Proposition 2.20. *Given any finite tensor product of quaternion algebras* $A := (a_1, b_1) \otimes \cdots \otimes (a_r, b_r)$*, there exists a quadratic form* q *of odd dimension* $2r+1$ *with trivial discriminant such that* $C_0(q) \simeq A$*.*

Remark 2.21. Thanks to Merkurjev's celebrated result [22], every element of ${}_2\mathrm{Br}(k)$ can be represented by a tensor product of quaternion algebras. Therefore, Proposition 2.20 implies that the assignment $q \mapsto [C_0(q)] \in {}_2\mathrm{Br}(k)$ is surjective.

Twisted quaternion projective spaces. We follow Example 1.5.

Theorem 2.22. *Given central simple* k*-algebras with involution of symplectic type* $(A, *)$ *and* $(A', *')$*, consider the following five conditions:*

(i) *we have* $[\mathrm{HP}(A, *)] = [\mathrm{HP}(A', *')]$ *in* $K_0\mathrm{Var}(k)$*;*
(ii) *we have* $\mu_{\mathrm{T}}([\mathrm{HP}(A, *)]) = \mu_{\mathrm{T}}([\mathrm{HP}(A', *')])$ *in* $R_{\mathrm{T}}(k)$*;*
(iii) *we have* $\deg(A) = \deg(A')$ *and an isomorphism* $A \simeq A'$*;*
(iv) *we have an isomorphism* $(A, *) \simeq (A', *')$ *of* k*-algebras with involution;*
(v) *we have* $\mathrm{HP}(A, *) \simeq \mathrm{HP}(A', *')$ *in* $\mathrm{Var}(k)$*.*

We have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftarrow$ (iv) $\Leftrightarrow$ (v) $\Rightarrow$ (i)*.*

Recall from [18, Thm. 3.1] that a central simple k-algebra A of even degree admits an involution $*$ of symplectic type if and only if $A \otimes A$ splits. Given any two such algebras A and A', with $A \not\simeq A'$, we hence conclude from the implication (i) $\Rightarrow$ (iii) of Theorem 2.22 that $[\mathrm{HP}(A, *)] \neq [\mathrm{HP}(A', *')]$ in $K_0\mathrm{Var}(k)$.

Involution varieties. We follow the notations/assumptions of Example 1.6.

Theorem 2.23. *Given central simple* k*-algebras with involution of orthogonal type* $(A, *)$ *and* $(A', *')$*, consider the following five conditions:*

(i) *we have* $[\mathrm{Iv}(A, *)] = [\mathrm{Iv}(A', *')]$ *in* $K_0\mathrm{Var}(k)$*;*
(ii) *we have* $\mu_{\mathrm{T}}([\mathrm{Iv}(A, *)]) = \mu_{\mathrm{T}}([\mathrm{Iv}(A', *')])$ *in* $R_{\mathrm{T}}(k)$*;*
(iii) *we have* $\deg(A) = \deg(A')$ *and isomorphisms*[4] $C_0^{\pm}(A, *) \simeq C_0^{\pm}(A', *')$*;*
(iv) *we have an isomorphism* $(A, *) \simeq (A', *')$ *of* k*-algebras with involution;*
(v) *we have* $\mathrm{Iv}(A, *) \simeq \mathrm{Iv}(A', *')$ *in* $\mathrm{Var}(k)$*.*

We have (i) $\Rightarrow$ (ii) $\Leftarrow$ (iii) $\Leftarrow$ (iv) $\Leftrightarrow$ (v) $\Rightarrow$ (i)*. Whenever* $\deg(A) = \deg(A') \neq 4$*, we have moreover the implication* (ii) $\Rightarrow$ (iii)*. This latter implication also holds when* A *(or* A'*) is a central division* k*-algebra of degree* 4*. Whenever* $\deg(A) = \deg(A')$ *is equal to* 4 *or* 6*, we have moreover the implication* (iii) $\Rightarrow$ (iv)*.*

Note that in the case where A and A' have degree 6, all the conditions (i)-(v) of Theorem 2.23 are equivalent. Therefore, two involution varieties, associated to central simple k-algebras of degree 6 with involution of orthogonal type, have the same Grothendieck class in $K_0\mathrm{Var}(k)$ if and only if they are isomorphic! To the best

[4] Concretely, we mean isomorphisms $C_0^+(A, *) \simeq C_0^+(A', *')$ and $C_0^-(A, *) \simeq C_0^-(A', *')$ or $C_0^+(A, *) \simeq C_0^-(A', *')$ and $C_0^-(A, *) \simeq C_0^+(A', *')$.

of the author's knowledge, this result is new in the literature. Recall from Remark 1.7 that this family of involution varieties contains all quadric hypersurfaces (2.15).

Note also that when A (or A') is a division k-algebra of degree 4, all the conditions (i)-(v) of Theorem 2.23 are equivalent. As proved by Albert in [1], every central simple k-algebra A of degree 4 and period 2 is isomorphic to a biquaternion k-algebra $(a_1, b_1)\otimes(a_2, b_2)$. It is moreover a division algebra if and only if the equation

$$a_1u^2 + b_1v^2 - a_1b_1w^2 - a_2x^2 - b_2y^2 + a_2b_2z^2 = 0\,, \tag{2.24}$$

in the variables u, v, w, x, y and z, has no non-trivial solutions; see [8, Thm. 1.5.5]. Thanks to the work of Tao [35, Thm. 4.15], we have $\mathrm{Iv}(A, *) \simeq C(a_1, b_1)\times C(a_2, b_2)$. Therefore, making use of Theorem 2.23, we obtain the refinement of Remark 2.6:

Corollary 2.25. *Given $a_1, b_1, a_2, b_2, a'_1, b'_1, a'_2, b'_2 \in k$, consider the two conditions:*
(i) *we have $[C(a_1, b_1) \times C(a_2, b_2)] = [C(a'_1, b'_1) \times C(a'_2, b'_2)]$ in $K_0\mathrm{Var}(k)$;*
(ii) *$C(a_1, b_1) \times C(a_2, b_2)$ is isomorphic to $C(a'_1, b'_1) \times C(a'_2, b'_2)$.*
We have (i) $\Leftarrow$ (ii). *Whenever the above equation* (2.24) *has no non-trivial solutions, we have moreover the implication* (i) $\Rightarrow$ (ii).

Example 2.26 (Rational functions of two variables). As explained in [19, §VI Examples 1.13 and 1.15], the equation (2.24) has no non-trivial solutions when:
(i) $k := \mathbb{R}(x, y)$, $a_1 := x$, $b_1 := -1$, $a_2 := -x$, and $b_2 := y$;
(ii) $k := \mathbb{Q}(x, y)$, $a_2 := x$, $b_2 := y$, and $a_1, b_1 \in \mathbb{Q}^\times$ are representatives of two independent square classes in $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$.
Making use of Corollary 2.25, we conclude that there is no difference between the Grothendieck and the isomorphism class of $C(a_1, b_1) \times C(a_2, b_2)$. Further examples exist for every field k with u-invariant equal to 6 or greater than 8; see [19, §XIII].

Remark 2.27. The quadric hypersurfaces (2.15) corresponding to Example 2.26(i)-(ii) are not birational to $\mathbb{P}^4$. Moreover, their Grothendieck classes are non-trivial.

3. Preliminaries

Throughout the article k denotes a base field.

Dg categories. Let $(\mathcal{C}(k), \otimes, k)$ be the category of (cochain) complexes of k-vector spaces. A *differential graded (=dg) category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$; consult Keller's ICM survey [12]. Every (dg) k-algebra A gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes since the category of perfect complexes $\mathrm{perf}(X)$ of every quasi-compact quasi-separated k-scheme X admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$. Let us denote by $\mathrm{dgcat}(k)$ the category of (small) dg categories.

Let $\mathcal{A}$ be a dg category. The opposite dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects as $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$. A *right dg $\mathcal{A}$-module* is a dg functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with values in the dg category $\mathcal{C}_{\mathrm{dg}}(k)$ of complexes of k-vector spaces. Let us denote by $\mathcal{C}(\mathcal{A})$ the category of right dg $\mathcal{A}$-modules. Following [12, §3.2], the *derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$* is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the objectwise quasi-isomorphisms. We write $\mathcal{D}_c(\mathcal{A})$ for the subcategory of compact objects.

A dg functor $F\colon \mathcal{A} \to \mathcal{B}$ is called a *derived Morita equivalence* if it induces an equivalence of categories $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{B})$; see [12, §4.6]. As proved in [32, Thm. 5.3], $\mathrm{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are the derived Morita equivalences. Let us denote by $\mathrm{Hmo}(k)$ the associated homotopy category.

The *tensor product* $\mathcal{A}\otimes\mathcal{B}$ of dg categories is defined as follows: the set of objects is the cartesian product and $(\mathcal{A}\otimes\mathcal{B})((x,w),(y,z)) := \mathcal{A}(x,y)\otimes\mathcal{B}(w,z)$. As explained in [12, §2.3], this construction gives rise to a symmetric monoidal structure on $\mathrm{dgcat}(k)$, which descends to the homotopy category $\mathrm{Hmo}(k)$.

A *dg $\mathcal{A}$-$\mathcal{B}$-bimodule* B is a dg functor $\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}\to\mathcal{C}_{\mathrm{dg}}(k)$ or equivalently a right dg $(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B})$-module. Associated to a dg functor $F\colon\mathcal{A}\to\mathcal{B}$, we have the dg $\mathcal{A}$-$\mathcal{B}$-bimodule ${}_F\mathcal{B}\colon\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}\to\mathcal{C}_{\mathrm{dg}}(k), (x,z)\mapsto\mathcal{B}(z,F(x))$. Let us write $\mathrm{rep}(\mathcal{A},\mathcal{B})$ for the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B})$ consisting of those dg $\mathcal{A}$-$\mathcal{B}$-bimodules B such that for every object $x\in\mathcal{A}$ the associated right dg $\mathcal{B}$-module $\mathrm{B}(x,-)$ belongs to $\mathcal{D}_c(\mathcal{B})$. Clearly, the dg $\mathcal{A}$-$\mathcal{B}$-bimodules ${}_F\mathcal{B}$ belongs to $\mathrm{rep}(\mathcal{A},\mathcal{B})$.

Following Kontsevich [15, 16, 17], a dg category $\mathcal{A}$ is called *smooth* if the dg $\mathcal{A}$-$\mathcal{A}$ bimodule ${}_{\mathrm{id}}\mathcal{A}$ belongs to $\mathcal{D}_c(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{A})$ and *proper* if $\sum_n \dim H^n\mathcal{A}(x,y)<\infty$ for any pair of objects (x,y). Examples include the finite dimensional k-algebras A of finite global dimension (when k is perfect) as well as the dg categories of perfect complexes $\mathrm{perf}_{\mathrm{dg}}(X)$ associated to smooth proper k-schemes X.

Noncommutative motives. For a recent book on noncommutative motives, we invite the reader to consult [30]. As proved in [32, Cor. 5.10], there is a natural bijection between $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A},\mathcal{B})$ and the set of isomorphism classes of the category $\mathrm{rep}(\mathcal{A},\mathcal{B})$. Under this bijection, the composition law corresponds to the tensor product of bimodules. The *additivization* of $\mathrm{Hmo}(k)$ is the additive category $\mathrm{Hmo}_0(k)$ with the same objects and with abelian groups of morphisms $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathcal{A},\mathcal{B})$ given by the Grothendieck group $K_0\mathrm{rep}(\mathcal{A},\mathcal{B})$ of the triangulated category $\mathrm{rep}(\mathcal{A},\mathcal{B})$. The composition law is induced by the tensor product of bimodules. Given a commutative ring of coefficients R, the *R-linearization* of $\mathrm{Hmo}_0(k)$ is the R-linear category $\mathrm{Hmo}_0(k)_R$ obtained by tensoring the morphisms of $\mathrm{Hmo}_0(k)$ with R. Note that we have the (composed) symmetric monoidal functor

$$U(-)_R\colon \mathrm{dgcat}(k)\longrightarrow \mathrm{Hmo}_0(k)_R \quad \mathcal{A}\mapsto\mathcal{A} \quad (\mathcal{A}\xrightarrow{F}\mathcal{B})\mapsto[{}_F\mathcal{B}]_R\,.$$

The category of *noncommutative Chow motives* $\mathrm{NChow}(k)_R$ is defined as the idempotent completion of the full subcategory of $\mathrm{Hmo}_0(k)_R$ consisting of the objects $U(\mathcal{A})_R$ with $\mathcal{A}$ a smooth proper dg category. This category is R-linear, additive, rigid symmetric monoidal, and idempotent complete. When $R=\mathbb{Z}$, we will write $\mathrm{NChow}(k)$ instead of $\mathrm{NChow}(k)_\mathbb{Z}$ and U instead of $U(-)_\mathbb{Z}$.

Notation 3.1. Let $\mathrm{CSA}(k)_R$ be the full subcategory of $\mathrm{NChow}(k)_R$ consisting of the objects $U(A)_R$ with A a central simple k-algebra. In the same vein, let $\mathrm{CSA}(k)_R^{\oplus}$ be the closure of $\mathrm{CSA}(k)_R$ under finite direct sums.

Given an additive rigid symmetric monoidal category $\mathcal{C}$, its *$\mathcal{N}$-ideal* is defined as

$$\mathcal{N}(a,b):=\{f\in\mathrm{Hom}_\mathcal{C}(a,b)\,|\,\forall g\in\mathrm{Hom}_\mathcal{C}(b,a) \text{ we have } \mathrm{tr}(g\circ f)=0\}\,,$$

where $\mathrm{tr}(g\circ f)$ stands for the categorical trace of the endomorphism $g\circ f$. The category of *noncommutative numerical motives* $\mathrm{NNum}(k)_R$ is defined as the idempotent completion of the quotient of $\mathrm{NChow}(k)_R$ by the $\otimes$-ideal $\mathcal{N}$. By construction, this category is R-linear, additive, rigid symmetric monoidal, and idempotent complete.

4. Proof of Theorem 1.8

Throughout this section k is a base field of characteristic zero. Let $K_0(\mathrm{NChow}(k))$ be the Grothendieck ring of the additive symmetric monoidal category of noncommutative Chow motives $\mathrm{NChow}(k)$. We start by constructing a motivic measure defined on the whole Grothendieck ring of varieties.

Proposition 4.1. *The assignment $X \mapsto U(\mathrm{perf}_{\mathrm{dg}}(X))$, with X a smooth projective k-scheme, gives rise to a motivic measure $\mu_{\mathrm{nc}} \colon K_0\mathrm{Var}(k) \to K_0(\mathrm{NChow}(k))$.*

Proof. Thanks to Bittner's presentation (see [4, Thm. 3.1]) of the Grothendieck ring of varieties $K_0\mathrm{Var}(k)$, it suffices to verify the following two conditions:

(i) given smooth projective k-schemes X and Y, we have the following equality:

$$[U(\mathrm{perf}_{\mathrm{dg}}(X \times Y))] = [U(\mathrm{perf}_{\mathrm{dg}}(X)) \otimes U(\mathrm{perf}_{\mathrm{dg}}(Y))]$$

in the Grothendieck ring $K_0(\mathrm{NChow}(k))$;

(ii) let X be a smooth projective k-scheme, $Y \hookrightarrow X$ a closed subscheme of codimension c, $\mathrm{Bl}_Y(X)$ the blow-up of X along Y, and E the exceptional divisor of the blow-up. Under these notations, we have the following equality

$$[U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Bl}_Y(X)))] - [U(\mathrm{perf}_{\mathrm{dg}}(E))] = [U(\mathrm{perf}_{\mathrm{dg}}(X))] - [U(\mathrm{perf}_{\mathrm{dg}}(Y))]$$

in the Grothendieck ring $K_0(\mathrm{NChow}(k))$.

The assignment $(\mathcal{G}, \mathcal{H}) \mapsto \mathcal{G} \boxtimes \mathcal{H}$ gives rise to a derived Morita equivalence between $\mathrm{perf}_{\mathrm{dg}}(X) \otimes \mathrm{perf}_{\mathrm{dg}}(Y)$ and $\mathrm{perf}_{\mathrm{dg}}(X \times Y)$. Therefore, condition (i) follows from the fact that the functor U is symmetric monoidal. In what concerns condition (ii), recall from Orlov [25, Thm. 4.3] that $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Bl}_Y(X))$ contains full dg subcategories $\mathrm{perf}_{\mathrm{dg}}(X), \mathrm{perf}_{\mathrm{dg}}(Y)_0, \ldots, \mathrm{perf}_{\mathrm{dg}}(Y)_{c-2}$ inducing a semi-orthogonal decomposition $\mathrm{perf}(\mathrm{Bl}_Y(X)) = \langle \mathrm{perf}(X), \mathrm{perf}(Y)_0, \ldots, \mathrm{perf}(Y)_{c-2}\rangle$. Moreover, $\mathrm{perf}_{\mathrm{dg}}(Y)_i$ is derived Morita equivalent to $\mathrm{perf}_{\mathrm{dg}}(Y)$ for every i. Making use of [32, Thm. 6.3(4)], which asserts that the functor U sends semi-orthogonal decomposition to direct sums, we hence obtain the following equality

$$[U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Bl}_Y(X)))] = [U(\mathrm{perf}_{\mathrm{dg}}(X))] + (c-1)[U(\mathrm{perf}_{\mathrm{dg}}(Y))] \tag{4.2}$$

in the Grothendieck ring $K_0(\mathrm{NChow}(k))$. Similarly, recall from Orlov [25, Thm. 2.6] that $\mathrm{perf}_{\mathrm{dg}}(E)$ contains full dg subcategories $\mathrm{perf}_{\mathrm{dg}}(Y)_0, \ldots, \mathrm{perf}_{\mathrm{dg}}(Y)_{c-1}$ inducing a semi-orthogonal decomposition $\mathrm{perf}(E) = \langle \mathrm{perf}(Y)_0, \ldots, \mathrm{perf}(Y)_{c-1}\rangle$. Moreover, $\mathrm{perf}_{\mathrm{dg}}(Y)_i$ is derived Morita equivalent to $\mathrm{perf}_{\mathrm{dg}}(Y)$ for every i. Making use once again of [32, Thm. 6.3(4)], we obtain the following equality

$$[U(\mathrm{perf}_{\mathrm{dg}}(E))] = c[U(\mathrm{perf}_{\mathrm{dg}}(Y))] \tag{4.3}$$

in the Grothendieck ring $K_0(\mathrm{NChow}(k))$. The proof of condition (ii) follows now from the combination of the above equalities (4.2)-(4.3). ∎

Recall from [34, Thm. 2.19(iv)] the following result:

Theorem 4.4. *Given central simple k-algebras $B_1, \ldots, B_n$ and $C_1, \ldots, C_m$, the following two conditions are equivalent:*

(i) *we have an isomorphism of noncommutative Chow motives*

$$U(B_1) \oplus \cdots \oplus U(B_n) \simeq U(C_1) \oplus \cdots \oplus U(C_m)\,;$$

(ii) *we have $n = m$ and for every prime number p there exists a permutation σ_p (which depends on p) such that $[C_i^p] = [B^p_{\sigma_p(i)}]$ in $\mathrm{Br}(k)\{p\}$ for every $1 \leq i \leq n$. Here, B^p and C^p stand for the p-primary components of B and C, respectively.*

The following result generalizes [34, Cor. 2.22].

Proposition 4.5. *Let $B_1, \ldots, B_n$ and $C_1, \ldots, C_m$ be central simple k-algebras, and $N\!M$ a noncommutative Chow motive. If $\oplus_{i=1}^n U(B_i) \oplus N\!M$ is isomorphic to $\oplus_{j=1}^m U(C_j) \oplus N\!M$ in $\mathrm{NChow}(k)$, then $n = m$ and $\oplus_{i=1}^n U(B_i) \simeq \oplus_{j=1}^n U(C_j)$.*

Proof. Given a (fixed) prime number p, consider the induced isomorphism

$$\oplus_{i=1}^n U(B_i)_{\mathbb{F}_p} \oplus N\!M_{\mathbb{F}_p} \simeq \oplus_{j=1}^m U(C_j)_{\mathbb{F}_p} \oplus N\!M_{\mathbb{F}_p} \tag{4.6}$$

in the category $\mathrm{NNum}(k)_{\mathbb{F}_p}$. Thanks to Lemma 4.9(i) below, there exist non-negative integers $r_i, 1 \leq i \leq n$, and $s_j, 1 \leq j \leq m$, and a noncommutative numerical motive $N\!M'$ such that $N\!M_{\mathbb{F}_p}$ and $\oplus_{i=1}^n U(B_i)_{\mathbb{F}_p}^{\oplus r_i} \oplus \oplus_{j=1}^m U(C_j)_{\mathbb{F}_p}^{\oplus s_j} \oplus N\!M'$ are isomorphic in $\mathrm{NNum}(k)_{\mathbb{F}_p}$. Consequently, (4.6) yields an induced isomorphism:

$$\oplus_{i=1}^n U(B_i)_{\mathbb{F}_p}^{\oplus (r_i+1)} \oplus \oplus_{j=1}^m U(C_j)_{\mathbb{F}_p}^{\oplus s_j} \oplus N\!M' \simeq \oplus_{i=1}^n U(B_i)_{\mathbb{F}_p}^{\oplus r_i} \oplus \oplus_{j=1}^m U(C_j)_{\mathbb{F}_p}^{\oplus (s_j+1)} \oplus N\!M'$$

We claim that the composition bilinear pairings (in $\mathrm{NNum}(k)_{\mathbb{F}_p}$)

$$\mathrm{Hom}(U(D)_{\mathbb{F}_p}, N\!M') \times \mathrm{Hom}(N\!M', U(E)_{\mathbb{F}_p}) \longrightarrow \mathrm{Hom}(U(D)_{\mathbb{F}_p}, U(E)_{\mathbb{F}_p}), \tag{4.7}$$

with $D, E \in \{U(B_1)_{\mathbb{F}_p}, \ldots, U(B_n)_{\mathbb{F}_p}, U(C_1)_{\mathbb{F}_p}, \ldots, U(C_m)_{\mathbb{F}_p}\}$, are all zero. On the one hand, if $p \mid \mathrm{ind}(D^{\mathrm{op}} \otimes E)$, then it follows from [33, Prop. 6.2(i)] that the right-hand side of (4.7) is zero. On the other hand, if $p \nmid \mathrm{ind}(D^{\mathrm{op}} \otimes E)$, then it follows from [33, Prop. 6.2(ii)] that $U(D)_{\mathbb{F}_p}$ is isomorphic to $U(E)_{\mathbb{F}_p}$. In this latter case, the right-hand side of (4.7) is isomorphic to $\mathbb{F}_p$. Since the category $\mathrm{NNum}(k)_{\mathbb{F}_p}$ is $\mathbb{F}_p$-linear, we then conclude from Lemma 4.9(ii) below that the bilinear pairing (4.7) is necessarily zero; otherwise the noncommutative numerical motive $N\!M'$ would contain $U(D)_{\mathbb{F}_p}$, or equivalently $U(E)_{\mathbb{F}_p}$, as a direct summand. Now, note that the triviality of the bilinear pairings (4.7) implies that the above isomorphism involving $N\!M'$ restricts to an isomorphism

$$\oplus_{i=1}^n U(B_i)_{\mathbb{F}_p}^{\oplus (r_i+1)} \oplus \oplus_{j=1}^m U(C_j)_{\mathbb{F}_p}^{\oplus s_j} \simeq \oplus_{i=1}^n U(B_i)_{\mathbb{F}_p}^{\oplus r_i} \oplus \oplus_{j=1}^m U(C_j)_{\mathbb{F}_p}^{\oplus (s_j+1)} \tag{4.8}$$

in the full subcategory $\mathrm{CSA}(k)_{\mathbb{F}_p}^{\oplus}/\mathcal{N}$ of $\mathrm{NNum}(k)_{\mathbb{F}_p}$. Recall that $\mathrm{Br}(k)\{p\}$ stands for the p-primary component of the Brauer group $\mathrm{Br}(k)$. As proved in [33, Prop. 6.11], the assignment $U(B)_{\mathbb{F}_p} \mapsto (\mathbb{F}_p)_{[B^p]}$ gives rise to an equivalence of categories between $\mathrm{CSA}(k)_{\mathbb{F}_p}^{\oplus}/\mathcal{N}$ and the category of $\mathrm{Br}(k)\{p\}$-graded finite dimensional $\mathbb{F}_p$-vector spaces. Since the latter category has the Krull-Schmidt property, it follows from the above isomorphism (4.8) that $\oplus_{i=1}^n U(B_i)_{\mathbb{F}_p} \simeq \oplus_{j=1}^m U(C_j)_{\mathbb{F}_p}$, that $m = n$, and that there exists a permutation σ_p (which depends on p) such that $[C_i^p] = [B^p_{\sigma_p(i)}] \in \mathrm{Br}(k)\{p\}$ for every $1 \leq i \leq n$. Finally, using the fact that the prime number p is arbitrary, we conclude from Theorem 4.4 that $\oplus_{i=1}^n U(B_i)$ is isomorphic to $\oplus_{i=1}^n U(C_i)$ in $\mathrm{NChow}(k)$. This finishes the proof. $\square$

Lemma 4.9. *There exist non-negative integers $r_i, 1 \leq i \leq n$, and $s_j, 1 \leq j \leq m$, and a noncommutative numerical motive $N\!M' \in \mathrm{NNum}(k)_{\mathbb{F}_p}$ such that:*

(i) *we have an isomorphism $N\!M_{\mathbb{F}_p} \simeq \oplus_{i=1}^n U(B_i)_{\mathbb{F}_p}^{\oplus r_i} \oplus \oplus_{j=1}^m U(C_j)_{\mathbb{F}_p}^{\oplus s_j} \oplus N\!M'$;*

(ii) *the noncommutative numerical motive $N\!M'$ does not contains the objects $U(B_i)_{\mathbb{F}_p}, 1 \leq i \leq n$, and $U(C_j)_{\mathbb{F}_p}, 1 \leq j \leq m$, as direct summands.*

Proof. The proof is similar to the one of [33, Lem. 6.17]. ◻

Let $K_0(\mathrm{CSA}(k)^{\oplus})$ be the Grothendieck ring of the category $\mathrm{CSA}(k)^{\oplus}$.

Proposition 4.10. (i) *The assignment $\sum_{i=1}^m b_i[B_i] \mapsto \sum_{i=1}^m b_i[U(B_i)]$ induces a ring isomorphism $R_{\mathrm{T}}(k) \xrightarrow{\sim} K_0(\mathrm{CSA}(k)^{\oplus})$;*

(ii) *The inclusion of categories $\mathrm{CSA}(k)^{\oplus} \subset \mathrm{NChow}(k)$ gives rise to an injective ring homomorphism $K_0(\mathrm{CSA}(k)^{\oplus}) \to K_0(\mathrm{NChow}(k))$.*

Proof. Let $K_0(\mathrm{NChow}(k))^+$ be the semi-ring of the additive symmetric monoidal category $\mathrm{NChow}(k)$. Concretely, $K_0(\mathrm{NChow}(k))^+$ is the set of isomorphism classes of noncommutative Chow motives equipped with the addition (resp. multiplication) law induced by $\oplus$ (resp. $\otimes$). In the same vein, let $K_0(\mathrm{CSA}(k)^{\oplus})^+$ be the semi-ring of the additive symmetric monoidal category $\mathrm{CSA}(k)^{\oplus}$.

Let B and C be two central simple k-algebras. As proved in [31, Thm. 9.1], $U(B)$ is isomorphic to $U(C)$ in $\mathrm{NChow}(k)$ if and only if $[B] = [C]$ in $\mathrm{Br}(k)$. Moreover, as proved in [34, Prop. 3.5], we have an isomorphism between $U(k) \oplus U(A \otimes B)$ and $U(A) \oplus U(B)$ in $\mathrm{CSA}(k)^{\oplus}$ whenever $(\mathrm{ind}(B), \mathrm{ind}(C)) = 1$. Consequently, by definition of the semi-ring $R_{\mathrm{T}}^+(k)$, the assignment $\sum_{i=1}^m b_i[B_i] \mapsto \oplus_{i=1}^m U(B_i)^{\oplus b_i}$ induces a surjective homomorphism $R_{\mathrm{T}}^+(k) \twoheadrightarrow K_0(\mathrm{CSA}(k)^{\oplus})^+$. Since the functor U is symmetric monoidal, it follows from [34, Prop. 3.10(ii)] that the latter homomorphism is moreover injective, and hence an isomorphism. The proof of item (i) follows now from the fact that $R_{\mathrm{T}}(k)$ and $K_0(\mathrm{CSA}(k)^{\oplus})$ are the group completions of $R_{\mathrm{T}}^+(k)$ and $K_0(\mathrm{CSA}(k)^{\oplus})^+$, respectively.

Given an arbitrary monoid $(M, +)$, recall that its group completion is defined as the quotient of $M \times M$ by the following equivalence relation:

$$(m,n) \sim (m',n') := \exists\, r \in M \text{ such that } m + n' + r = n + m' + r\,. \tag{4.11}$$

Since the inclusion $\mathrm{CSA}(k)^{\oplus} \subset \mathrm{NChow}(k)$ gives rise to an injective homomorphism $K_0(\mathrm{CSA}(k)^{\oplus})^+ \to K_0(\mathrm{NChow}(k))^+$, the preceding definition (4.11) combined with Proposition 4.5 allows us to conclude that the induced (ring) homomorphism $K_0(\mathrm{CSA}(k)^{\oplus}) \to K_0(\mathrm{NChow}(k))$ is also injective. This proves item (ii). ◻

Consider the (composed) ring homomorphism

$$K_0\mathrm{Var}(k)^{\mathrm{tw}} \subset K_0\mathrm{Var}(k) \xrightarrow{\mu_{\mathrm{nc}}} K_0(\mathrm{NChow}(k))\,. \tag{4.12}$$

Thanks to Proposition 4.10, we have also the injective ring homomorphism

$$R_{\mathrm{T}}(k) \longrightarrow K_0(\mathrm{NChow}(k)) \qquad \sum_{i=1}^m b_i[B_i] \mapsto \sum_{i=1}^m b_i[U(B_i)]\,. \tag{4.13}$$

Let ${}_\gamma\mathcal{F}$ be a twisted projective homogeneous variety. As proved in [31, Thm. 2.1] (with $E = U$), we have an isomorphism between $U(\mathrm{perf}_{\mathrm{dg}}({}_\gamma\mathcal{F}))$ and $\oplus_{i=1}^{n(\mathcal{F})} U(A_i)$ in $\mathrm{NChow}(k)$. Since the Grothendieck class of the noncommutative Chow motive $\oplus_{i=1}^{n(\mathcal{F})} U(A_i)$ agrees with the image of $\sum_{i=1}^{n(\mathcal{F})} [A_i]$ under the injective ring homomorphism (4.13), we then conclude that (4.12) takes values in the subring $R_{\mathrm{T}}(k)$. In other words, (4.12) yields the searched ring homomorphism $\mu_{\mathrm{T}}\colon K_0\mathrm{Var}(k)^{\mathrm{tw}} \to R_{\mathrm{T}}(k), [{}_\gamma\mathcal{F}] \mapsto \sum_{i=1}^{n(\mathcal{F})} [A_i]$. This finishes the proof of Theorem 1.8.

5. Proof of Proposition 1.11

As explained in the proof of Proposition 4.10, we have the semi-ring isomorphism

$$(5.1) \qquad R^+_{\mathrm{T}}(k) \xrightarrow{\sim} K_0(\mathrm{CSA}(k)^{\oplus})^+ \qquad \sum_{i=1}^m b_i[B_i] \mapsto \oplus_{i=1}^m U(B_i)^{\oplus b_i}\,.$$

Moreover, $R^+_{\mathrm{T}}(k)$ and $K_0(\mathrm{CSA}(k)^{\oplus})$ are the group completions of the semi-rings $R^+_{\mathrm{T}}(k)$ and $K_0(\mathrm{CSA}(k)^{\oplus})^+$, respectively. In order to prove item (i), it suffices then to show that the homomorphism $K_0(\mathrm{CSA}(k)^{\oplus})^+ \to K_0(\mathrm{CSA}(k)^{\oplus})$ is injective. Thanks to Definition 4.11, this follows from Proposition 4.5 (with $N\!M \in \mathrm{CSA}(k)^{\oplus}$).

Recall that the index and the period of a central simple k-algebra have the same prime factors. Therefore, by combining the p-primary decomposition of the Brauer group $\mathrm{Br}(k) = \oplus_p \mathrm{Br}(k)\{p\}$ with Theorem 4.4, we conclude that the following map

$$K_0(\mathrm{CSA}(k)^{\oplus})^+ \longrightarrow \{\text{subgroups of } \mathrm{Br}(k)\} \quad \oplus_{i=1}^m U(B_i)^{\oplus b_i} \mapsto \langle [B_1], \ldots, [B_m]\rangle$$

is well-defined. By pre-composing it with (5.1), we hence obtain the searched map $\sum_{i=1}^m b_i[B_i] \mapsto \langle [B_1], \ldots, [B_m]\rangle$. Note that, by construction, the latter map is surjective. This proves item (ii).

6. Proof of Proposition 2.10

Assume first that the period of A and A' is $2, 3, 4$ or 6. Thanks to the implication (i) $\Rightarrow$ (iii) of Theorem 2.1, if $[\mathrm{SB}(A)] = [\mathrm{SB}(B)]$ in $K_0\mathrm{Var}(k)$, then $\deg(A) = \deg(A')$ and $\langle [A]\rangle = \langle [A']\rangle$. This latter equivalence implies that the Brauer class $[A]$ is equal to $[A']$ or $-[A']$. If $[A] = [A']$, the Severi-Brauer varieties $\mathrm{SB}(A)$ and $\mathrm{SB}(A')$ are isomorphic (and hence birational). In $[A] = -[A']$, Amitsur's conjecture holds thanks to the work of Roquette [28].

Assume now that the central simple k-algebras A and A' have period 5 and even degree. Note that the equality $\langle [A]\rangle = \langle [A']\rangle$ implies that the Brauer class $[A]$ is equal to $[A']$, $-[A']$, $2[A']$, or $-2[A']$. In the first two cases, the above arguments apply. If $[A] = 2[A']$, Amitsur's conjecture holds thanks to the work of Tregub [37]. If $[A] = -2[A']$, the combination of the works of Roquette and Tregub allows us also to conclude that $\mathrm{SB}(A)$ and $\mathrm{SB}(B)$ are birational. This finishes the proof.

7. Proof of Theorem 2.12

We start with the following result of independent interest:

Proposition 7.1. *Given twisted projective homogeneous varieties ${}_\gamma\mathcal{F}$ and ${}_{\gamma'}\mathcal{F}'$, the following two conditions are equivalent:*
(a) *we have $\mu_{\mathrm{T}}([{}_\gamma\mathcal{F}]) = \mu_{\mathrm{T}}([{}_{\gamma'}\mathcal{F}'])$ in $R_{\mathrm{T}}(k)$;*
(b) *we have an isomorphism $U(\mathrm{perf}_{\mathrm{dg}}({}_\gamma\mathcal{F})) \simeq U(\mathrm{perf}_{\mathrm{dg}}({}_{\gamma'}\mathcal{F}'))$ in $\mathrm{NChow}(k)$.*

Proof. By construction of the Tits' motivic measure μ_{T} (see §4), if condition (a) holds, then $[U(\mathrm{perf}_{\mathrm{dg}}({}_\gamma\mathcal{F}))]$ is equal to $[U(\mathrm{perf}_{\mathrm{dg}}({}_{\gamma'}\mathcal{F}'))]$ in $K_0(\mathrm{NChow}(k))$. By definition of the Grothendieck ring $K_0(\mathrm{NChow}(k))$, there exists then a noncommutative Chow motive $N\!M$ such that $U(\mathrm{perf}_{\mathrm{dg}}({}_\gamma\mathcal{F})) \oplus N\!M$ and $U(\mathrm{perf}_{\mathrm{dg}}({}_{\gamma'}\mathcal{F}')) \oplus N\!M$ are isomorphic in $\mathrm{NChow}(k)$. Using the fact that $U(\mathrm{perf}_{\mathrm{dg}}({}_\gamma\mathcal{F}))$ is isomorphic to the direct sum $\oplus_{i=1}^{n(\mathcal{F})} U(A_i)$, where $A_1, \ldots, A_{n(\mathcal{F})}$ are the Tits' algebras of ${}_\gamma\mathcal{F}$, we hence conclude from Proposition 4.5 that noncommutative Chow motives $U(\mathrm{perf}_{\mathrm{dg}}({}_\gamma\mathcal{F}))$ and $U(\mathrm{perf}_{\mathrm{dg}}({}_{\gamma'}\mathcal{F}'))$ are isomorphic. This shows (a) $\Rightarrow$ (b).

If condition (b) holds, then $[U(\mathrm{perf}_{\mathrm{dg}}({}_\gamma\mathcal{F}))]$ is equal to $[U(\mathrm{perf}_{\mathrm{dg}}({}_{\gamma'}\mathcal{F}'))]$ in the Grothendieck ring $K_0(\mathrm{NChow}(k))$. By construction of the Tits' motivic measure μ_{T}, this implies condition (a). □

Remark 7.2 (Products of twisted projective homogeneous varieties). Given smooth projective k-schemes X and Y, the assignment $(\mathcal{G},\mathcal{H})\mapsto \mathcal{G}\boxtimes\mathcal{H}$ gives rise to a derived Morita equivalence between $\mathrm{perf}_{\mathrm{dg}}(X\times Y)$ and $\mathrm{perf}_{\mathrm{dg}}(X)\otimes\mathrm{perf}_{\mathrm{dg}}(Y)$. Since the functor U is symmetric monoidal, we hence conclude that the noncommutative Chow motives $U(\mathrm{perf}_{\mathrm{dg}}(X\times Y))$ and $U(\mathrm{perf}_{\mathrm{dg}}(X))\otimes U(\mathrm{perf}_{\mathrm{dg}}(Y))$ are isomorphic to $\mathrm{NChow}(k)$. This implies that Proposition 7.1 holds similarly with ${}_\gamma\mathcal{F}$ and ${}_{\gamma'}\mathcal{F}'$ replaced by products of twisted projective homogeneous varieties.

The implication (i) $\Rightarrow$ (ii) follows automatically from Theorem 1.8, and the implications (iv) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (i) are clear.

Let us now prove the equivalence (ii) $\Leftrightarrow$ (iii). Assume condition (ii). As explained in Example 1.2, the motivic measure (1.9) sends the Grothendieck class $[\mathrm{Gr}(d;A)]$ to $\binom{\deg(A)}{d}$. Therefore, we have the equality $\binom{\deg(A)}{d}=\binom{\deg(A')}{d'}$. By definition of the binomial coefficient, this implies that $\deg(A)=\deg(A')$ and that $d'=d_A$ or $d'=\deg(A)-d$. By combining Theorem 1.8 with Proposition 1.11, we conclude moreover that $\langle[A]\rangle=\langle[A']\rangle$. This shows the implication (ii) $\Rightarrow$ (iii). Assume now condition (iii). If $d'=d$, then [34, Thm. 3.20] implies that $U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Gr}(d;A)))$ is isomorphic to $U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Gr}(d';A')))$ in $\mathrm{NChow}(k)$. If $d'=\deg(A)-d$, run the same proof using the equality of Gaussian polynomials $\binom{\deg(A)}{\deg(A)-d}_t=\binom{\deg(A)}{d}_t$ in the variable t. Making use of Proposition 7.1, we hence conclude that $\mu_{\mathrm{T}}([\mathrm{Gr}(d;A)])$ is equal to $\mu_{\mathrm{T}}([\mathrm{Gr}(d';A')])$ in $R_{\mathrm{T}}(k)$. This shows the implication (iii) $\Rightarrow$ (ii).

Let us now prove the equivalence (iv) $\Leftrightarrow$ (v). The implication (iv) $\Rightarrow$ (v) is clear when $d'=d$ and follows from the canonical isomorphism between $\mathrm{Gr}(d;A)$ and $\mathrm{Gr}(\deg(A)-d;A)$ when $d'=\deg(A)-d$. Assume now condition (v). Making use of the implication (v) $\Rightarrow$ (iii) (proved above), we conclude that $\deg(A)=\deg(A')$ and that $d'=d$ or $d'=\deg(A)-d$. If $d'=d$, then the one-to-one correspondence $A\mapsto\mathrm{Gr}(d;A)$ between central simple k-algebras and twisted Grassmannian varieties implies that $A\simeq A'$. If $d'=\deg(A)-d$, the isomorphism between $\mathrm{Gr}(d;A)$ and $\mathrm{Gr}(\deg(A)-d;A)$ and the one-to-one correspondence $A\mapsto\mathrm{Gr}(\deg(A)-d;A)$ between central simple k-algebras and twisted Grassmannian varieties implies also that $A\simeq A'$. This shows the implication (v) $\Rightarrow$ (iv).

Finally, assume condition (iii) and that A and A' have period 2. Under these assumptions, we have $\langle[A]\rangle=\langle[A']\rangle$ if and only if $[A]=[A']$ in $\mathrm{Br}(k)$. Therefore, the proof of the implication (iii) $\Rightarrow$ (iv) follows from the fact that two central simple k-algebras with the same degree and Brauer class are necessarily isomorphic.

8. Proof of Proposition 2.13

Recall from Example 1.2 that the motivic measure (1.9) sends $[\mathrm{Gr}(d;A)]$ to $\binom{\deg(A)}{d}$. Therefore, if $[\mathrm{Gr}(d;A)\times\mathrm{Gr}(d;A')]$ agrees with $[\mathrm{Gr}(d'';A'')\times\mathrm{Gr}(d'';A''')]$ in $K_0\mathrm{Var}(k)$, we have $\binom{\deg(A)}{d}^2=\binom{\deg(A'')}{d''}^2$. Consequently, we obtain the equality $\binom{\deg(A)}{d}=\binom{\deg(A'')}{d''}$. By definition of the binomial coefficient, this implies that $\deg(A)=\deg(A'')$ and that $d''=d$ or $d''=\deg(A)-d$. Making use of Theorem 1.8 and Proposition 1.11, we conclude moreover that $\langle[A],[A']\rangle=\langle[A''],[A''']\rangle$.

If by hypothesis A, A', A'', A''' have period 2, then the Brauer class $[A]$ is necessarily equal to $[A'']$, $[A''']$, or $[A''\otimes A''']$. In the first two cases, we conclude from the implication (iii) $\Rightarrow$ (v) of Theorem 2.12 that $\mathrm{Gr}(d; A)$ is isomorphic to $\mathrm{Gr}(d''; A'')$ or $\mathrm{Gr}(d''; A''')$. In the remaining case, since $\langle [A], [A']\rangle = \langle [A''], [A''']\rangle$, the Brauer class $[A']$ is necessarily equal to $[A'']$ or $[A''']$. Making use once again of the implication (iii) $\Rightarrow$ (v) of Theorem 2.12, we conclude that $\mathrm{Gr}(d; A')$ is isomorphic to $\mathrm{Gr}(d''; A'')$ or $\mathrm{Gr}(d''; A''')$. This finishes the proof.

9. Proof of Theorem 2.14

The implication (i) $\Rightarrow$ (ii) follows from Theorem 1.8, the implication (iv) $\Rightarrow$ (iii) and equivalence (iv) $\Leftrightarrow$ (v) are well-known, and the implication (v) $\Rightarrow$ (i) is clear.

Let us now prove the equivalence (ii) $\Leftrightarrow$ (iii). Assume condition (ii). As explained in Example 1.4, the motivic measure (1.9) sends the Grothendieck class $[Q_q]$ to $\dim(q)-1$, resp. $\dim(q)$, in the odd-dimensional, resp. even-dimensional, case. Therefore, we have the equality $\dim(q)=\dim(q')$. By combining Theorem 1.8 with Proposition 1.11, we conclude moreover that $\langle [C_0(q)]\rangle = \langle [C_0(q')]\rangle$, resp. $\langle [C_0^+(q)], [C_0^-(q)]\rangle = \langle [C_0^+(q)], [C_0^-(q')]\rangle$, in the odd-dimensional, resp. even-dimensional, case. Since the even Clifford algebras $C_0(q), C_0^+(q), C_0^-(q)$ have period 2 and $C_0^+(q)\simeq C_0^-(q)$, this implies that $[C_0(q)] = [C_0(q')]$, resp. $[C_0^+(q)] = [C_0^+(q)]$, in the odd-dimensional, resp. even-dimensional, case. Using the fact that $\deg(C_0(q)) = 2^{[\frac{\dim(q)}{2}]}$, resp. $\deg(C_0^+(q)) = 2^{\frac{\dim(q)}{2}-1}$, in the odd-dimensional, resp. even-dimensional, case, we conclude that $C_0(q)\simeq C_0(q')$, resp. $C_0^+(q)\simeq C_0^+(q')$, in the odd-dimensional, resp. even-dimensional, case. This shows the implication (ii) $\Rightarrow$ (iii). Assume now condition (iii). As explained in [31, Example 3.8] (with $E=U$), the noncommutative Chow motive $U(\mathrm{perf}_{\mathrm{dg}}(Q_q))$ is isomorphic to

$$U(k)^{\oplus \dim(q)-2}\oplus U(C_0(q)) \qquad \text{resp. } U(k)^{\oplus \dim(q)-2}\oplus U(C_0^+(q))\oplus U(C_0^-(q))$$

in the odd-dimensional, resp. even-dimensional, case. Therefore, if by hypothesis $\dim(q)=\dim(q')$ and $C_0(q)\simeq C_0(q')$, resp. $C_0^+(q)\simeq C_0^+(q')$, in the odd-dimensional, resp. even-dimensional, case, we conclude that the noncommutative Chow motives $U(\mathrm{perf}_{\mathrm{dg}}(Q_q))$ and $U(\mathrm{perf}_{\mathrm{dg}}(Q_{q'}))$ are isomorphic. Thanks to Proposition 7.1, this implies condition (ii).

Finally, assume condition (iii) and that $\dim(q)=\dim(q')$ is equal to 3 or 6. Recall from [18, Thm. 15.2], resp. [18, Cor. 15.33], that the assignment $q\mapsto C_0(q)$, resp. $q\mapsto C_0^+(q)$, induces a one-to-one correspondence between similarity classes of quadratic forms of dimension 3, resp. dimension 6, with trivial discriminant and isomorphism classes of quaternion algebras, resp. central simple k-algebras of degree 4 and period 2 (=biquaternion algebras). Making use of these latter correspondences, we hence conclude that the quadratic forms q and q' are similar. This shows the implication (iii) $\Rightarrow$ (iv).

Remark 9.1 (Criterion for motivic equivalence). Let q and q' be two quadratic forms of odd dimension. By combining Proposition 7.1 with the equivalence (ii) $\Leftrightarrow$ (iii) of Theorem 2.14, we observe that the associated noncommutative Chow motives $U(\mathrm{perf}_{\mathrm{dg}}(Q_q))$ and $U(\mathrm{perf}_{\mathrm{dg}}(Q_{q'}))$ are isomorphic if and only if $\dim(q)=\dim(q')$ and $C_0(q)\simeq C_0(q')$. In the case of Chow motives, Vishik [38] and Izhboldin [10, 11] proved that the Chow motives $\mathfrak{h}(Q_q)$ and $\mathfrak{h}(Q_{q'})$ are isomorphic if and only if the quadratic forms q and q' are similar. This shows that the criterion for

motivic equivalence in the commutative world is much more restrictive than the corresponding criterion in the noncommutative world.

10. Proof of Proposition 2.18

The implication (ii) $\Rightarrow$ (i) is clear. Recall from the proof of Theorem 2.14 that the noncommutative Chow motive $U(\mathrm{perf}_{\mathrm{dg}}(Q_q))$ is isomorphic to the direct sum $U(k)^{\oplus 4} \oplus U(C_0^+(q)) \oplus U(C_0^-(q))$. Making use of the k-algebra isomorphisms $C_0^+(q) \simeq C_0^-(q)$ and $C_0^+(q') \simeq C_0^-(q')$, of the derived Morita equivalence $(\mathcal{G}, \mathcal{H}) \mapsto \mathcal{G} \boxtimes \mathcal{H}$ between $\mathrm{perf}_{\mathrm{dg}}(Q_q) \otimes \mathrm{perf}_{\mathrm{dg}}(Q_{q'})$ and $\mathrm{perf}_{\mathrm{dg}}(Q_q \times Q_{q'})$, and of the fact that the functor U is symmetric monoidal, we hence conclude that the noncommutative Chow motive $U(\mathrm{perf}_{\mathrm{dg}}(Q_q \times Q_{q'}))$ is isomorphic to

$$U(k)^{\oplus 16} \oplus U(C_0^+(q))^{\oplus 8} \oplus U(C_0^+(q'))^{\oplus 8} \oplus U(C_0^+(q) \otimes C_0^+(q'))^{\oplus 4}\,. \tag{10.1}$$

Assume now condition (i). Under this assumption, $\mu_{\mathrm{T}}([Q_q \times Q_{q'}])$ is equal to $\mu_{\mathrm{T}}([Q_{q''} \times Q_{q'''}])$ in $R_{\mathrm{T}}(k)$. Thanks to Proposition 7.1 and Remark 7.2, this implies that $U(\mathrm{perf}_{\mathrm{dg}}(Q_q \times Q_{q'}))$ and $U(\mathrm{perf}_{\mathrm{dg}}(Q_{q''} \times Q_{q'''}))$ are isomorphic in $\mathrm{NChow}(k)$. Consider the following (distinct and exhaustive) four cases:

(a) assume that $[C_0^+(q)] = [C_0^+(q')] = [k]$. In this case, it follows from the combination of (10.1) with Theorem 4.4 that $[C_0^+(q)] = \cdots = [C_0^+(q''')] = [k]$;

(b) assume that $[C_0^+(q)] \neq [k]$ and $[C_0^+(q')] = [k]$ (or the converse). In this case, we have $[C_0^+(q) \otimes C_0^+(q')] = [C_0^+(q)]$. Consequently, since the computation (10.1) yields 24 copies of $[k]$ and 12 copies of $[C_0^+(q')]$, it follows from Theorem 4.4 that $[C_0^+(q)] = [C_0^+(q'')]$ and $[C_0^+(q')] = [C_0^+(q''')] = [k]$ (or vice-versa);

(c) assume that $[k] \neq [C_0^+(q)] = [C_0^+(q')] \neq [k]$. In this case, $[C_0^+(q) \otimes C_0^+(q')] = [k]$. Consequently, since the computation (10.1) yields 20 copies of $[k]$ and 16 copies of $[C_0^+(q)]$, it follows from Theorem 4.4 that $[C_0^+(q)] = \cdots = [C_0^+(q''')]$;

(d) assume that $[k] \neq [C_0^+(q)] \neq [C_0^+(q')] \neq [k]$. In this case, $[C_0^+(q) \otimes C_0^+(q')]$ is different from $[k]$, $[C_0^+(q)]$, and $[C_0^+(q')]$. Consequently, since the computation (10.1) yields 16 copies of $[k]$, 8 copies of $[C_0^+(q)]$, 8 copies of $[C_0^+(q')]$, and 4 copies of $[C_0^+(q) \otimes C_0^+(q')]$, it follows from Theorem 4.4 that $[C_0^+(q)] = [C_0^+(q'')]$ and $[C_0^+(q')] = [C_0^+(q''')]$ (or vice-versa).

Since the Clifford algebras $C_0^+(q), \ldots, C_0^+(q''')$ have degree 4, it follows from (a)-(d) that $C_0^+(q) \simeq C_0^+(q'')$ and $C_0^+(q') \simeq C_0^+(q''')$ (or vice-versa). Making use of the implication (iii) $\Rightarrow$ (v) of Theorem 2.14, we hence conclude that $Q_q \simeq Q_{q''}$ and $Q_{q'} \simeq Q_{q'''}$ (or vice-versa). In both cases, $Q_q \times Q_{q''}$ is isomorphic to $Q_{q''} \times Q_{q'''}$. This shows the implication (i) $\Rightarrow$ (ii).

11. Proof of Proposition 2.20

Recall from the proof of Theorem 2.14 that the assignment $q' \mapsto C_0(q')$ induces a one-to-one correspondence between similarity classes of quadratic forms of dimension 3 with trivial discriminant and isomorphism classes of quaternion algebras. Let us denote by q_1 the quadratic form of dimension 3 corresponding to (a_1, b_1). Consider also the following quadratic forms of dimension 2

$$q_2 := -\langle a_2, b_2 \rangle \qquad q_j := (-1)^{j-1}(a_2 b_2 \cdots a_{j-1} b_{j-1})\langle a_j, b_j \rangle \quad 3 \leq j \leq r$$

and the orthogonal sum $q := q_1 \perp \cdots \perp q_r$ of dimension $2r+1$. Now, an (increasing) inductive argument using the following natural identifications (see [19, §V])

$$C_0(q' \perp q'') \simeq C_0(q') \otimes C(-\delta(q') \cdot q'') \quad C(\langle a_j, b_j \rangle) \simeq (a_j, b_j) \quad \delta(\langle a_j, b_j \rangle) = -a_j b_j\,,$$

where q' is an odd-dimensional quadratic form and $C(-)$ stands for the Clifford algebra construction, allows us to conclude that $C_0(q) \simeq A$. This finishes the proof.

12. Proof of Theorem 2.22

The implication (i) $\Rightarrow$ (ii) follows from Theorem 1.8, the implications (iv) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (i) are clear, and the equivalence (iv) $\Leftrightarrow$ (v) is well-known.

Let us now prove the equivalence (ii) $\Leftrightarrow$ (iii). Assume condition (ii). As explained in Example 1.5, the motivic measure (1.9) sends the Grothendieck class $[\mathrm{HP}(A,*)]$ to $\frac{\deg(A)}{2}$. Therefore, we have the equality $\deg(A) = \deg(A')$. By combining Theorem 1.8 with Proposition 1.11, we conclude moreover that $\langle [A]\rangle = \langle [A']\rangle$. Since the central simple k-algebras A and A' admits an involution (of symplectic type), they have period 2. This implies that $[A] = [A']$. Using the fact that $\deg(A) = \deg(A')$, we hence conclude that $A \simeq A'$. This shows the implication (ii) $\Rightarrow$ (iii). Assume now condition (iii). By combining Ananyevskiy's description of the representation ring $R(Sp_2 \times Sp_{n-2})$ (see [3, §5.3 and §9.3]) with [31, Thm. 2.1(i)] (with $E = U$), we obtain the following computation in the category of noncommutative Chow motives

$$U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{HP}(A,*))) \simeq U(k)^{\oplus \lceil \frac{\deg(A)}{4} \rceil} \oplus U(A)^{\oplus \lfloor \frac{\deg(A)}{4} \rfloor},$$

where $\lceil\ \rceil$ and $\lfloor\ \rfloor$ stand for the ceiling and floor functions, respectively. Therefore, if by hypothesis $\deg(A) = \deg(A')$ and $A \simeq A'$, we conclude that the noncommutative Chow motives $U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{HP}(A,*)))$ and $U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{HP}(A',*')))$ are isomorphic. Thanks to Proposition 7.1, this implies condition (ii).

13. Proof of Theorem 2.23

The implication (i) $\Rightarrow$ (ii) follows from Theorem 1.8, the implications (iv) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (i) are clear, and the equivalence (iv) $\Leftrightarrow$ (v) is well-known.

Let us now prove the implication (iii) $\Rightarrow$ (ii). Assume condition (iii). As explained in [31, Example 3.11] (with $E = U$), $U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Iv}(A,*)))$ is isomorphic to

$$U(k)^{\oplus \frac{\deg(A)}{2} - 1} \oplus U(A)^{\oplus \frac{\deg(A)}{2} - 1} \oplus U(C_0^+(A,*)) \oplus U(C_0^-(A,*)). \tag{13.1}$$

When $\deg(A) \equiv 2 \pmod 4$, we have the following relations

$$2[C_0^+(A,*)] = [A] \quad 3[C_0^+(A,*)] = [C_0^-(A,*)] \quad 4[C_0^+(A,*)] = [k] \tag{13.2}$$

in the Brauer group $\mathrm{Br}(k)$; see [18, §9.C]. When $\deg(A) \equiv 0 \pmod 4$, we have instead the following relations:

$$2[C_0^+(A,*)] = [k] \quad 2[C_0^-(A,*)] = [k] \quad [C_0^+(A,*)] + [C_0^-(A,*)] = [A]. \tag{13.3}$$

Note that if $C_0^{\pm}(A,*) \simeq C_0^{\pm}(A',*')$, then the preceding relations (13.2)-(13.3) imply that $[A] = [A']$. Since by assumption $\deg(A) = \deg(A')$, we hence conclude from the combination of the computation (13.1) with Theorem 4.4 that the noncommutative Chow motives $U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Iv}(A,*)))$ and $U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Iv}(A',*')))$ are isomorphic. Thanks to Proposition 7.1, this implies condition (ii).

Let us now prove implication (ii) $\Rightarrow$ (iii). Assume condition (ii). Recall from Proposition 7.1 that this latter condition is equivalent to the fact that the noncommutative Chow motives $U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Iv}(A,*)))$ and $U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Iv}(A',*')))$ are isomorphic. Since $[A]$, $[C_0^+(A,*)]$ and $[C_0^-(A,*)]$ belong to $\mathrm{Br}(k)\{2\}$, the above computation (13.1), combined with Theorem 4.4, implies that $\deg(A) = \deg(A')$ and that

the following sets of Brauer classes (containing $\frac{\deg(A)}{2}-1$ copies of $[k]$, $[A]$ and $[A']$)

$$\begin{aligned} &\{[k],\ldots,[k],[A],\ldots,[A],[C_0^+(A,*)],[C_0^-(A,*)]\} \\ &\{[k],\ldots,[k],[A'],\ldots,[A'],[C_0^+(A',*')],[C_0^-(A',*')]\} \end{aligned} \tag{13.4}$$

are the same up to permutation. When $\deg(A) \equiv 2 \pmod 4$, we hence conclude from the relations (13.2) that $[C_0^\pm(A,*)] = [C_0^\pm(A',*')]$. Using the equalities $\deg(C_0^\pm(A,*)) = 2^{\frac{\deg(A)}{2}-1}$, this implies that $C_0^\pm(A,*) \simeq C_0^\pm(A',*')$. When $\deg(A) \equiv 0 \pmod 4$ and $\deg(A) \neq 4$, we conclude from the relations (13.3) and from the fact that (13.4) contains at least 3 copies of $[A]$ that $[C_0^\pm(A,*)] = [C_0^\pm(A',*')]$. Using the equalities $\deg(C_0^\pm(A,*)) = 2^{\frac{\deg(A)}{2}-1}$, this implies that $C_0^\pm(A,*) \simeq C_0^\pm(A',*')$. Finally, when $\deg(A) = 4$ and A is a central division k-algebra, we conclude from the relations (13.3) that $[C_0^\pm(A,*)] = [C_0^\pm(A',*')]$; note that $[A]$ is necessarily different from $[C_0^\pm(A',*')]$ because $\mathrm{ind}(A) = 4 \neq 2 = \mathrm{ind}(C_0^\pm(A',*'))$.

Finally, assume condition (iii) and that $\deg(A) = \deg(A')$ is equal to 4 or 6. Consider the groupoid ${}^1\mathsf{D}_2$ of central simple k-algebras of degree 4 with involution of orthogonal type and trivial discriminant. In the same vein, let ${}^1\mathsf{A}_1^2$ be the groupoid of k-algebras of the form $B \times C$ with B and C quaternion algebras; the morphisms are the k-algebra isomorphisms. As proved in [18, Cor. 15.12], the assignment $(A,*) \mapsto C_0^+(A,*) \times C_0^-(A,*)$ gives rise to an equivalence of groupoids ${}^1\mathsf{D}_2 \overset{\simeq}{\to} {}^1\mathsf{A}_1^2$. Making use of the isomorphisms $C_0^\pm(A,*) \simeq C_0^\pm(A',*')$, we hence obtain from the preceding equivalence of groupoids an isomorphism $(A,*) \simeq (A',*')$ of k-algebras with involution. This shows condition (iv) in the case where $\deg(A) = 4$. Now, consider the ${}^1\mathsf{D}_3$ of central simple k-algebras of degree 6 with involution of orthogonal type and trivial discriminant. In the same vein, let ${}^1\mathsf{A}_3$ be the groupoid of pairs $(B \times B^{\mathrm{op}}, \epsilon)$ where B is a central simple k-algebra of degree 4 and ϵ the exchange involution; the morphisms are the k-algebra isomorphisms that preserve the involution. As proved in [18, Cor. 15.32], the assignment $(A,*) \mapsto (C_0^+(A,*) \times C_0^-(A,*), \epsilon)$ gives rise to an equivalence of groupoids ${}^1\mathsf{D}_3 \overset{\simeq}{\to} {}^1\mathsf{A}_3$. Moreover, as explained in *loc. cit.*, the objects $(C_0^+(A,*) \times C_0^-(A,*), \epsilon)$ and $(C_0^-(A,*) \times C_0^+(A,*), \epsilon)$ are isomorphic. Making use of the isomorphisms $C_0^\pm(A,*) \simeq C_0^\pm(A',*')$, we hence obtain from the preceding equivalence of groupoids an isomorphism $(A,*) \simeq (A',*')$ of k-algebras with involution. This shows condition (iv) in the case where $\deg(A) = 6$.

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